High School Math Contest

University of South Carolina

January 30th, 2016

Instructions

1. Do not open the booklet until you are told to do so.
2. This is a thirty (30) question multiple-choice test with questions on six (6) pages. Each question is followed by answers marked (a), (b), (c), (d), and (e). Exactly one of these answers is correct for each question.
3. The test will be scored as follows: Five (5) points for each correct answer, one (1) point for each answer that is left blank, and zero (0) points for each incorrect answer.
4. Your teacher should have provided you with a SCANTRON sheet. Fill in your 5-digit registration number from your name tag. For cross-reference, also print your name and school on the blank names labeled NAME and TEST, respectively.
5. Using a #2 pencil, record your answers on the SCANTRON sheet. **Do not make any further marks on the SCANTRON sheet except for your answers**. Use only the first 30 lines of the side of the form where you filled in your registration information. Record your answers with heavy marks and, if you make corrections, erase thoroughly. Use a blank sheet of paper to cover your answers as you work through the test.
6. No calculators or reference material are allowed[1]. Test proctors and judges have been instructed not to elaborate on any specific test question. All electronic devices must be turned off and put out of sight. Any electronic device seen during the exam will be confiscated and returned at the end of the test.
7. Drawings on the test are not necessarily drawn to scale.
8. When you are given the signal, begin working the problems. You have 90 minutes working time for the test.
9. If your booklet has a defective, missing, or illegible page, please notify the proctor in your room as soon as possible.
10. You must stay in the room for the entire exam period. If you need to use the restroom before the end of the exam period, quietly notify a proctor. Be sure to completely cover all exam materials before getting up from your desk. Return directly to your testing room.
11. Work quietly until the 90 minute time period has elapsed. Return the SCANTRON sheet and pencil to the proctor. You may keep this booklet. (Solutions will be posted online at `www.math.sc.edu/contest/`). Unless instructed otherwise by your teachers, take your belongings and meet your teachers in the first floor lobby.

Notation

- When n is a positive integer, $n!$ denotes the product $1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. For example, $3! = 1 \cdot 2 \cdot 3 = 6$.
- We denote by $\{1, 2, 3, \ldots, 99, 100\}$ the set of positive integers from 1 to 100.
- We denote by $\overline{AB}$ the line segment with endpoints A and B, and we denote by AB the length of the line segment $\overline{AB}$.
- For a real number x, we denote by $\lfloor x \rfloor$ (the floor) the largest integer which is less than or equal to x. We denote by $\lceil x \rceil$ (the ceiling) the smallest integer which is greater than or equal to x.

[1]Unless specifically approved in advance by Dr. Blanco-Silva.

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Problem 1. The sum of two prime numbers is 85. What is the product of these two prime numbers?

(a) 85 (b) 91 (c) 115 (d) 133 (e) 166

Problem 2. A group of people, cats, and birds has seventy legs, thirty heads, and twenty tails. How many cats are among this group? (Assume all birds have two legs and a tail.)

(a) 0 (b) 5 (c) 10 (d) 15 (e) 20

Problem 3. A 2-digit number is such that the product of the digits plus the sum of the digits is equal to the number. What is the units digit of the number?

(a) 1 (b) 3 (c) 5 (d) 7 (e) 9

Problem 4. What is $\log_3(4) \times \log_4(5) \times \cdots \times \log_{80}(81)$?

(a) 1 (b) 4 (c) 81 (d) $\log_{80} 243$ (e) $e^{81} - 1$

Problem 5. You walk one mile to school every day. You leave home at the same time each day, walk at a steady speed of 3 miles per hour, and arrive just as school begins. Today you were distracted by the pleasant weather and walked the first half mile at a speed of only 2 miles per hour. At how many miles per hour must you run the last half mile in order to arrive just as school begins today?

(a) 4 (b) 6 (c) 8 (d) 10 (e) 12

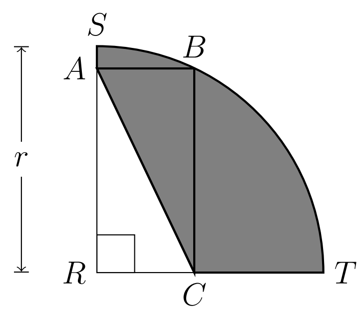
Problem 6. An ice cream cone is three inches tall and its top is a circle with diameter two inches. The cone is filled with ice cream, such that the interior of the cone is entirely full. The cone is topped with a hemisphere of ice cream with diameter two inches. What is the total volume, in cubic inches, of the ice cream in and atop the cone?

(a) π (b) $\frac{4}{3}\pi$ (c) $\frac{3}{2}\pi$ (d) $\frac{5}{3}\pi$ (e) 2π

Problem 7. In the Clock Game, part of the game show *The Price Is Right*, a contestant must guess the price (rounded to the nearest dollar) of a prize which is worth less than $2,000. After each guess, the contestant is told whether her guess was correct, too low, or too high. Assume that the contestant is mathematically savvy but has no idea how much the prize is worth. With how many guesses is she guaranteed to win the prize?

(a) 10 (b) 11 (c) 12 (d) 13 (e) 1997

Problem 8. In the figure below, arc SBT is one quarter of a circle with center R and radius r. The length plus the width of rectangle $ABCR$ is 8, and the perimeter of the shaded region is $10 + 3\pi$. Find the value of r.



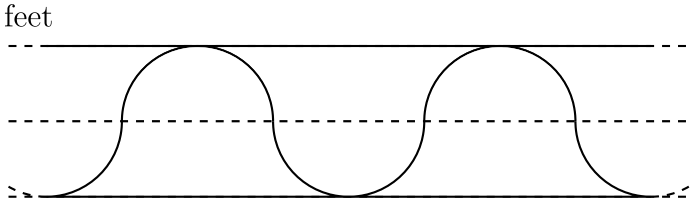
(a) 6 (b) 6.25 (c) 6.5 (d) 6.75 (e) 7

Problem 9. One urn contains two pieces of candy–one green and one red. A second urn contains four pieces of candy–one green and three red. For each urn, each piece of candy is equally likely of being picked. You pick a piece of candy from each urn and eat the two chosen candies.

If you eat exactly one piece of green candy, you draw a second piece of candy from the urn still containing a green piece of candy. You now eat the candy you just chose. What is the probability that you ate two pieces of green candy?

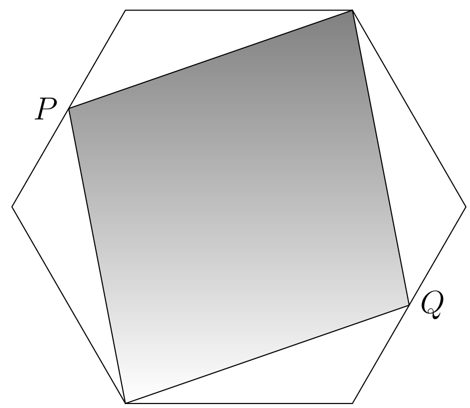
(a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{3}{8}$ (d) $\frac{1}{2}$ (e) $\frac{5}{8}$

Problem 10. A straight one-mile stretch of highway is 40 feet wide. You ride your bike on a path composed of semicircles as shown. If you ride at 5 miles per hour, how many hours will it take you to cover the one-mile stretch?
Note: 1 mile = 5280 feet



(a) $\frac{\pi}{11}$ (b) $\frac{\pi}{10}$ (c) $\frac{\pi}{5}$ (d) $\frac{2\pi}{5}$ (e) $\frac{2\pi}{3}$

Problem 11. In the diagram below, the points P and Q are the midpoints of opposite sides of a regular hexagon. What fraction of the hexagon is shaded?



(a) 2/3 (b) 3/4 (c) 5/6 (d) 7/8 (e) 11/12

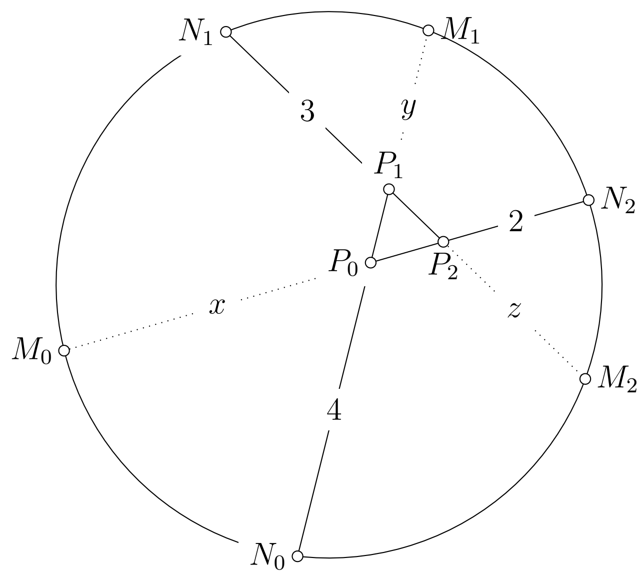
Problem 12. One day in December 2015, three 2-digit prime numbers A, B and C, were given to three members of a High School math team: Ashley, Beth, and Caitlin (respectively). They had this conversation:
Ashley: "If you two add your numbers, we get precisely today's date!"
Beth: "If you two add your numbers, we get my birthday this month, which was before today."
Caitlin: "If you two add your numbers, we get my birthday this month, which is after today."
What number did Caitlin get?

(a) 11 (b) 13 (c) 17 (d) 19 (e) 23

Problem 13. Every day, the value of a stock rises by exactly two dollars in the morning, and then falls by exactly one dollar in the afternoon. If the stock's value at the start of the first day is \$100, on what day will the stock's value first reach \$200?

(a) 50 (b) 99 (c) 100 (d) 101 (e) 200

Problem 14. In the diagram below, a triangle $\triangle P_0P_1P_2$ is equilateral with side-length 1. The length of segment $\overline{P_0N_0}$ is 4, the length of segment $\overline{P_1N_1}$ is 3, and the length of segment $\overline{P_2N_2}$ is 2. Let $x = P_0M_0$, $y = P_1M_1$, and $z = P_2M_2$. Compute $x + y + z$.



(a) 6 (b) 7 (c) 8 (d) 9 (e) 10

Problem 15. Seven different playing cards, with values from ace to seven, are shuffled and placed in a row on a table to form a seven-digit number. What is the probability that this seven-digit number is divisible by 11?
Note: Each of the possible seven-digit numbers is equally likely to occur.

(a) 4/35 (b) 1/7 (c) 8/35 (d) 2/7 (e) 12/35

Problem 16. Given the sequence $\{x_n\}$ defined by $x_{n+1} = \frac{1+x_n\sqrt{3}}{\sqrt{3}-x_n}$ with $x_1 = 1$, compute the value of $x_{2016} - x_{618}$.

(a) 0 (b) 1 (c) $\sqrt{3}$ (d) 2 (e) $2\sqrt{3}$

Problem 17. Compute the minimum value of the expression $\sin^{2016}\alpha + \cos^{2016}\alpha$, for $\alpha \in \mathbb{R}$.

(a) 2^{-1007} (b) 2^{-1006} (c) 2^{-1005} (d) 2^{-1004} (e) 2^{-1003}

Problem 18. Suppose $f(x)$ is an odd function for which $f(x + 2) = f(x)$ for all x, and $f(x) = x^2$ for $x \in (0, 1)$. Compute $f(-3/2) + f(1)$.

(a) -1 (b) $-\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) $\frac{1}{4}$ (e) $\frac{1}{2}$

Problem 19. Consider the sequence $\{a_n\}$ given by $a_{n+1} = \arctan(\sec a_n)$, with $a_1 = \frac{\pi}{6}$. Find the value of a positive integer m that satisfies $\sin a_1 \sin a_2 \cdots \sin a_m = \frac{1}{100}$.

(a) 333 (b) 334 (c) 666 (d) 3333 (e) 6666

Problem 20. The polynomial $x^4 - 27x^2 + 121$ can be factored in a unique way into a product of two quadratic polynomials with integer coefficients and leading coefficient 1. What is the sum of these two polynomials?

(a) $2x^2 - 5x + 122$ (b) $2x^2 - 5x - 22$ (c) $2x^2 - 5x + 22$ (d) $2x^2 - 22$ (e) $2x^2 + 22$

Problem 21. You choose at random ten points inside of a circle, so that no two of them are on any diameter. What is the probability that the circle has some diameter, so that exactly five points are on one side and exactly five points are on the other?

(a) $\frac{63}{256}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{5}{6}$ (e) 1

Problem 22. Let $x_1, x_2, \ldots, x_k$ be the distinct real solutions of the equation $x^3 - 3\lfloor x \rfloor = 4$. Compute $x_1^3 + x_2^3 + \cdots + x_k^3$.

(a) 13 (b) 14 (c) 15 (d) 16 (e) 17

Problem 23. Let $\alpha_1, \ldots, \alpha_k$ be the distinct real numbers in $[0, 2\pi]$ which satisfy the equation $\sin x + \cos x = -1/3$. What is the value of $\alpha_1 + \cdots + \alpha_k$?

(a) $7\pi/4$ (b) 2π (c) $9\pi/4$ (d) $5\pi/2$ (e) $11\pi/4$

Problem 24. If n is a positive integer, write $s(n)$ for the sum of the digits of n. What is $s(1) + s(2) + s(3) + \cdots + s(1000)$?

(a) 9991 (b) 10000 (c) 13501 (d) 14999 (e) 15000

Problem 25. Consider an infinite supply of cardboard equilateral triangles and squares, all of them with side length of one inch. What would be the convex polygon (without holes) with the largest number of sides that you could construct with these pieces, if the pieces are not allowed to overlap?

(a) hexagon (b) octogon (c) 10-gon (d) 12-gon (e) 16-gon

Problem 26. Two positive numbers a and b satisfy $2 + \log_2 a = 3 + \log_3 b = \log_6(a + b)$. Compute the value of $\frac{1}{a} + \frac{1}{b}$.

(a) 2 (b) 3 (c) 108 (d) 216 (e) 324

Problem 27. Consider the sequence $\{a_n\}$ given by $a_{n+1} = 2(n+2)a_n/(n+1)$, with $a_1 = 2$. Compute the value of

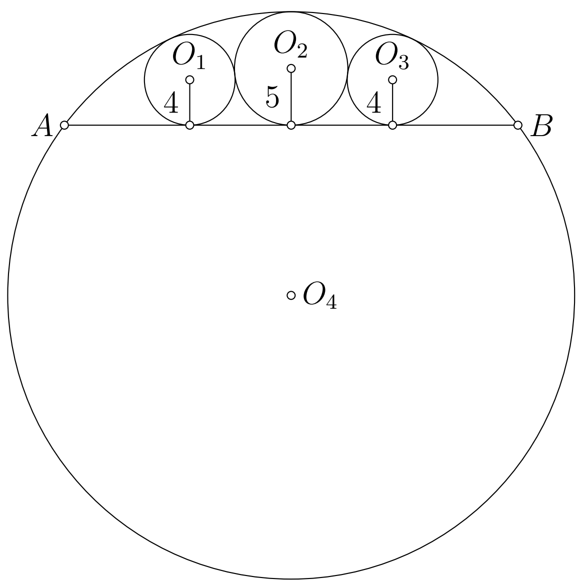
$$\frac{a_{2016}}{a_1 + a_2 + \cdots + a_{2015}}.$$

(a) $\frac{2016}{2015}$ (b) $\frac{2016}{2014}$ (c) $\frac{2017}{2016}$ (d) $\frac{2017}{2015}$ (e) $\frac{2017}{2014}$

Problem 28. Let $x = 0.82^{0.5}$, $y = \sin 1$ (radians), and $z = \log_3 \sqrt{7}$. Which of the following statements is true?

(a) $x < y < z$ (b) $y < z < x$ (c) $y < x < z$ (d) $z < y < x$ (e) $z < x < y$

Problem 29. The figure below shows four tangent circles: C_1 with center O_1 and radius 4, C_2 with center O_2 and radius 5, C_3 with center O_3 and radius 4, and C_4 with center O_4r. Chord $\overline{AB}$ of circle C_4 is tangent to circles C_1, C_2 and C_3. Find AB.



(a) 30 (b) 32 (c) 35 (d) 37 (e) 40

Problem 30. Find $\arctan(1/3) + \arctan(1/5) + \arctan(1/7) + \arctan(1/8)$.

(a) $\pi/6$ (b) $\pi/5$ (c) $\pi/4$ (d) $\pi/3$ (e) $\pi/2$